MIFP-06-37

hep-th/0612293

Group Reduction of Heterotic Supergravity

H. Lü, C.N. Pope and E. Sezgin

‡ *George P. and Cynthia W. Mitchell Institute for Fundamental Physics,*

Texas A&M University, College Station, TX 77843, USA

ABSTRACT

The reduction of ten-dimensional heterotic supergravity with Yang-Mills symmetry group K is performed on an arbitrary n-dimensional group manifold G. The reduction involves a nonvanishing 3-form flux, and the Lie algebra of G must have traceless structure constants to ensure the consistency of the reduction at the level of the action. A large class of gauged supergravities in $d = 10 - n$ with (non)compact gaugings is obtained. The resulting models describe half-maximal gauged supergravities coupled to $(n + \dim K)$ vector multiplets. We uncover their hidden $SO(n, n + \dim K)$ duality symmetry, and the $SO(n, n + \dim K)/SO(n) \times SO(n + \dim K)$ coset structure that governs the couplings of the scalar fields. We find that the local gauge symmetry of the d-dimensional theory is $K \times G \ltimes R^n$. Differences from the existing gauged supergravities are highlighted. The consistent truncation to pure half-maximal gauged supergravity in any dimension is shown, and the obstacle to performing a chiral truncation of the theory in $d = 6$ dimensions is found. Among the results obtained are the complete diagonalisation of the fermionic kinetic terms, and other reduction formulae that are applicable to group reductions of supergravities in arbitrary dimensions.

Contents

1 Introduction

In this paper, we study the group manifold "DeWitt reduction" of ten-dimensional heterotic supergravity. More specifically, we study the reduction of the theory on a group manifold G of dimension n. The group need not be compact or simple, but the structure constants of the underlying Lie algebra must be traceless to ensure consistent reduction at the level of the action [1].

This work grew out of our attempts to understand the string/M-theory origin of the remarkable anomaly-free gauged supergravities that exist in six dimensions [2, 3]. Models of this type have been increasingly finding applications in cosmology, and in braneworld scenarios [4–9]. One of the anomaly-free models, which has $E_7 \times E_6 \times U(1)_R$ symmetry, was found long ago [10]. More recently anomaly-free models with $E_7 \times G_2 \times U(1)_R$ symmetry [11] and $F_4 \times Sp(9) \times U(1)_R$ symmetry [12] have been found.

Progress has been made in embedding a minimal sub-sector with $U(1)_R$ symmetry and no hyperfermions in string/M theory [13]. This result has been generalized to some extent, to include a larger set of fields, in [14]. In both of these efforts a key role is played by noncompact gaugings in half-maximal supergravities in $d = 7$ coupled to a number of vector multiplets. In particular, an $SO(2, 2)$ gauged theory is reduced on a circle to $d = 6$ and then chirally truncated to obtain the desired result. In [13] it was argued that the theory in $d = 7$ can itself be obtained via a consistent reduction of heterotic supergravity on a certain 3-manifold $H_{2,2}$, which is a hyperboloid embedded in R^4. In [14], on the other hand, certain noncompact gauged theories were directly considered in $d = 7$, and their chiral reduction to $d = 6$ was obtained. Among these models were an $SO(2, 1)$ gauged half-maximal supergravity coupled to a single vector multiplet. Given that gauged supergravities are known to result from a DeWitt (or Scherk-Schwarz) reduction on a group manifold, it became natural to consider the prospects of obtaining the $SO(2, 1)$ gauged model from the group reduction of the heterotic supergravity theory. This has been the prime motivation for the results which will be reported in this paper.

To begin with, we find the full bosonic Lagrangian in d dimensions that results from the reduction on the group manifold G. It describes the coupling of half-maximal supergravity to $(n + \dim K)$ vector multiplets. The reduced theory is invariant under local $K \times G \ltimes R^n$ transformations. Here, G is any n-dimensional group based on a Lie algebra with traceless structure constants. Such algebras are sometimes referred to as type A, to distinguish them from those whose structure constants have a non-vanishing trace, which are referred to as type B. It is necessary to work with type A Lie algebras in order to ensure that the group reduction is consistent when the ansatz is substituted into the higher-dimensional action, as was observed in [1]. By contrast, a type B reduction works consistently only at the level of the field equations. Although substitution of the reduction ansatz for a type B algebra into the higher-dimensional equations of motion consistently yields equations of motion for the lower-dimensional theory, these equations cannot themselves be derived from an action. This was discovered first in [15], in the study of homogeneous cosmological models, and was more recently utilized in [16] in the group reduction of maximal supergravities.

A complete list of Lie algebras up to dimension five has been given in [17], although we shall use the notation of the list provided in [18]. There are 12 three-dimensional algebras, 20 four-dimensional algebras and 40 five-dimensional algebras that are not themselves direct sums of lower-dimensional algebras. Of these, there are 5, 4 and 19 respectively that are isolated type A Lie algebras. In addition, there are 2 examples in dimension four, and 4 examples in dimension five, that have a non-trivial free parameter, and a further 3 examples in dimension five that are two-parameter families of Lie algebras. These algebras with non-trivial parameters introduce new arbitrary parameters in the gauged supergravities obtained by the group reduction.

We also perform the group reduction of the supersymmetry transformations, up to cubic order in fermions. In doing so, we derive general formulae that can be used in the group reductions of a large class of supergravity theories in diverse dimensions. In particular, we obtain a complete diagonalisation of all the fermion kinetic terms. This goes beyond what has generally been done in the earlier literature.

Next, we exhibit the hidden $SO(n, n + \dim K)$ duality symmetry of the d-dimensional theory, and the $SO(n, n + \dim K)/SO(n) \times SO(\dim K)$ coset structure that governs the couplings of the scalar fields. We do so by first setting the gauge couplings to zero, thereby reducing the models to half-maximal supergravity coupled to $(n + K)$ Maxwell multiplets. This symmetry structure has already been exhibited in [19] at the level of the bosonic action, in the context of toroidal reduction. Here we generalize those results to include the supersymmetry transformation rules. We then generalize further by turning on the gauge couplings, finding dramatically simplified formulae for the action and supersymmetry transformation rules in an arbitrary dimension d. In particular, the potential takes a simple and universal form with a transparent group-theoretical structure. Thus, we have embedded a large class of gauge theories in heterotic string theory.

Turning to the issue of the consistent truncation of these theories, we have found that $(n + K)$ vector multiplets can always be truncated consistently to yield a pure half-maximal gauged supergravity theory in $d = 10-n$ dimensions, in cases where the n-dimensional gauge group is compact, gauging the R symmetry.

By half-maximal supergravities we mean those with 16 real supersymmetries, and gauging refers to the R-symmetry group. Such supergravities exist in $d \leq 8$ and they have been constructed directly in a given dimension over the years, and there are scattered results on particular group manifold reductions to obtain a small subset of them. Half-maximal gauged supergravities in dimensions $3 \leq d \leq 8$ can be found in [20–32] and there exists some results on group manifold reductions that yields a small set of such theories [16, 33–36].

Finally, we have examined the question that originally motivated our work, namely the prospects for obtaining either the $SO(2,1)$ gauged theory of [14], or a direct chiral truncation in $d = 6$ with surviving gauged R-symmetry. We find that neither the $SO(2,1)$ model nor the R-symmetry gauged chiral $d = 6$ model can be obtained from the models we have obtained by the group reduction of the heterotic supergravity, even if the group manifold G is taken to be $SO(2,1)$. The obstruction in $d = 7$ is due to the fact that the gauge symmetry of the model we have obtained is $K \times SO(2,1) \ltimes R^3$, and the truncation of two vector multiplets that is needed in order to obtain the $SO(2,1)$ gauged model of [14] is not possible since they are charged under the $SO(2,1)$ symmetry. The nonexistence of the chiral truncation in $d = 6$ is also closely related to this fact, and it is explained in detail in Section 5.2.

The paper is organized as follows. In the next section, the ten-dimensional heterotic supergravity model is presented in our notation and conventions, followed by the group reduction ansatz for all the fields. The Scherk-Schwarz formulae are generalized by turning on the 3-form flux. In Section 3, the full bosonic Lagrangian and supersymmetry transformation rules, up to cubic fermions, are obtained. The hidden symmetries are uncovered in Section 4, and the consistent truncations, or possible obstructions to them, are described in Section 5. Further comments on our results are collected in the conclusions. An appendix contains a table listing of the relevant Lie algebras considered in this paper, of dimensions 2, 3 and 4.

2 Ten-Dimensional Heterotic Supergravity

The Lagrangian, up to quartic fermion terms, is given by [37, 38]

$$\begin{aligned} \mathcal{L} &= \mathcal{L}_B + \mathcal{L}_F \qquad (2.1) \\ \mathcal{L}_B &= R{*}\mathbb{1} - \tfrac{1}{2}{*}d\phi \wedge d\phi - \tfrac{1}{2}e^{\hat{a}\phi}{*}G \wedge G - \tfrac{1}{2}e^{\hat{a}\phi/2}\,\mathrm{tr}'{*}F \wedge F \end{aligned}$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_F &= -\bar\psi_M\Gamma^{MNP}D_N\psi_P - \tfrac12\bar\chi\Gamma^M D_M\chi - \tfrac12\mathrm{tr}'\bar\lambda\Gamma^M D_M\lambda - \tfrac12\bar\psi_M\Gamma^N\Gamma^M\chi\partial\phi \\
&\quad +\tfrac14 e^{\hat a\phi/2}G_{PQR}\left(\bar\psi^M\Gamma_{[M}\Gamma^{PQR}\Gamma_{N]}\psi^N - \bar\psi_M\Gamma^{PQR}\Gamma^M\chi + \tfrac12\mathrm{tr}'\bar\lambda\Gamma^{PQR}\lambda\right) \\
&\quad -\tfrac12 e^{\hat a\phi/4}\mathrm{tr}'F_{MN}\left(\bar\lambda\Gamma^P\Gamma^{MN}\psi_P + \tfrac12\bar\lambda\Gamma^{MN}\chi\right)\ ,
\end{aligned} \tag{2.2}
$$

where the field strengths are defined as

$$
G = dB - \tfrac12\omega_{3Y}\ ,\qquad \omega_{3Y} = \mathrm{tr}'(FA - \tfrac13 A^3)\ ,\qquad F = dA + A^2\ ,\qquad D = d + [A,\ ]\ . \tag{2.3}
$$

Note that the $\chi^2 G$ type couplings which arise in dimensions $D < 10$ are absent in ten dimensions. In $D = 10$, we have

$$
\hat a = -1\ , \tag{2.4}
$$

but we shall define it as

$$
\hat a^2 = \frac{8}{D-2}\ , \tag{2.5}
$$

since then the reduction formula we obtain may be used when starting from an arbitrary dimension D. We have also used the notation $A = A^I T_I$, $A^2 = A\wedge A$, $FA = F\wedge A$, etc. The generators are anti-hermitian, obeying the Lie algebra $[T_I, T_J] = f^K{}_{IJ}T_J$, and normalized *in the fundamental representation* as

$$
\mathrm{tr}\, T_I T_J = \beta'\,\delta_{IJ}\ ,\qquad \beta' = \begin{cases} -1 & \text{for}\quad SO(32) \\ -30 & \text{for}\quad E_8^i\ ,\quad i = 1,2 \end{cases} \tag{2.6}
$$

We have also defined the normalized trace in the fundamental representation as

$$
\mathrm{tr}' = \tfrac{1}{\beta'}\,\mathrm{tr}\ . \tag{2.7}
$$

Note that, both for $SO(32)$ as well as E_8, we have $\mathrm{tr}'\, T_I T_J = \delta_{IJ}$.

The supersymmetry transformations, up to terms cubic in fermion, are:

$$
\begin{aligned}
\delta e_M^a &= \tfrac12\bar\epsilon\Gamma^a\psi_M \\
\delta\phi &= -\tfrac12\bar\epsilon\chi \\
\delta B_{MN} &= e^{-\hat a\phi/2}\left(\bar\epsilon\Gamma_{[M}\psi_{N]} + \tfrac{\hat a}{4}\,\bar\epsilon\Gamma_{MN}\chi\right) - \mathrm{tr}'A_{[M}\delta A_{N]} \\
\delta A_M &= \tfrac12 e^{-\hat a\phi/4}\bar\epsilon\Gamma_M\lambda \\
\delta\psi_M &= D_M\epsilon + \tfrac{\hat a^2}{96}\,e^{\hat a\phi/2}\left(\Gamma_M\Gamma^{PQR} - 12\delta_M^P\Gamma^{QR}\right)G_{PQR}\epsilon \\
\delta\chi &= -\tfrac12\Gamma^M\partial_M\phi\epsilon + \tfrac{1}{24}e^{\hat a\phi/2}\Gamma^{MNP}G_{MNP}\epsilon \\
\delta\lambda &= -\tfrac12 e^{\hat a\phi/4}\Gamma^{MN}F_{MN}\epsilon
\end{aligned} \tag{2.8}
$$

For completeness, we also record the Green-Schwarz anomaly counterterm and the modified Bianchi identities. The former can be read of from the anomaly polynomial [39]

$$\Omega_{12} = -\frac{1}{48 \times (2\pi)^6} X_4 X_8 \ , \tag{2.9}$$

where

$$\begin{aligned} X_4 &= \mathrm{tr}F^2 - \mathrm{tr}R^2 \ , \\ X_8 &= \mathrm{tr}F^4 - \tfrac{1}{8}\mathrm{tr}F^2\mathrm{tr}R^2 + \tfrac{1}{32}(\mathrm{tr}R^2)^2 + \tfrac{1}{8}\mathrm{tr}R^4 \ , \end{aligned} \tag{2.10}$$

for $SO(32)$ and

$$\begin{aligned} X_4 &= \tfrac{1}{30}\mathrm{tr}F_1^2 + \tfrac{1}{30}\mathrm{tr}F_2^2 - \mathrm{tr}R^2 \ , \\ X_8 &= \tfrac{1}{3600}\left[(\mathrm{tr}F_1^2)^2 + (\mathrm{tr}F_2^2)^2 - \mathrm{tr}F_1^2\mathrm{tr}F_2^2\right] \\ &\quad -\tfrac{1}{240}\left[\mathrm{tr}F_1^2 + \mathrm{tr}F_2^2\right]\mathrm{tr}R^2 + \tfrac{1}{32}(\mathrm{tr}R^2)^2 + \tfrac{1}{8}\mathrm{tr}R^4 \ , \end{aligned} \tag{2.11}$$

for $E_8 \times E_8$. We use the notation: $R = \frac{1}{2}dx^M dx^N R_{MN}{}^{ab}(\frac{1}{2}T_{ab})$ with the anti-hermitian generators $T_{ab}^{cd} = \delta_a^c\delta_b^d - \delta_a^d\delta_b^c$.

The anomalies can be read off from Ω_{12} via the descent equations, and they are cancelled by adding the Green-Schwarz counterterm

$$\mathcal{L}_{GS} = \frac{1}{48 \times (2\pi)^6}\left(BX_8 + \tfrac{2}{3}X_3^0 X_7^0\right) \ , \tag{2.12}$$

where X_3^0 and X_7^0 are defined by $X_4 = dX_3^0$ and $X_8 = dX_7^0$, respectively, together with a modification the 3-form field strength in which the Lorentz Chern-Simons term is added:

$$G \quad \to \quad G + \omega_{3L} \ . \tag{2.13}$$

In what follows, we shall not reduce the Green-Schwarz counterterm to lower dimensions, since the reduced theories we shall obtain will not have any perturbative anomalies, owing to the fact that they are either in odd dimensions, where there are no chiral fermions, or else in even dimensions but always vector-like. Should any chiral truncation be possible in an even lower dimension, the anomaly counterterms might play a role and they would have to be considered. Reduction of the anomaly counterterms would also make sense if all the Kaluza-Klein modes were to be kept, which amounts to re-writing the original theory in the lower-dimensional language. As for the Lorentz Chern-Simons modifications of the transformation rules, we shall not reduce them either, since they represent α' corrections to the self-contained lowest-order heterotic supergravity.

2.1 The reduction ansatz

In this section we shall use to a large extent the notation and some of the results of [40].

<u>*The Metric*:</u>

We begin with the reduction ansatz for the metric:

$$d\hat{s}^2 = e^{2\alpha\varphi}\, ds^2 + e^{2\beta\varphi}\, h_{\alpha\beta}\, \nu^\alpha\, \nu^\beta \ , \tag{2.14}$$

where

$$\nu^\alpha \equiv \sigma^\alpha - \mathcal{A}^\alpha \ . \tag{2.15}$$

Here ds^2 is the metric in d dimensions, A^α are the Yang-Mills potentials for the gauge group G, and $h_{\alpha\beta}$ is a unimodular symmetric matrix parameterising the scalar degrees of freedom. The group manifold is n-dimensional, and thus we have

$$D = d + n \ . \tag{2.16}$$

The Yang-Mills field strengths are given by

$$F^\alpha = d\mathcal{A}^\alpha + \tfrac{1}{2} f^\alpha{}_{\beta\gamma}\, \mathcal{A}^\beta \wedge A^\gamma \ , \tag{2.17}$$

and the constants α and β are chosen to be

$$\alpha = -\sqrt{\frac{n}{2(d-2)(D-2)}} \ , \qquad \beta = -\frac{\alpha(n-2)}{n} \ . \tag{2.18}$$

These choices ensure that the reduction of the Einstein-Hilbert action from D dimensions yields a pure Einstein-Hilbert term, and that φ has a canonically-normalised kinetic term, in d dimensions.

We shall choose the vielbein basis to be

$$\hat{e}^a = e^{\alpha\varphi}\, e^a \ , \qquad \hat{e}^i = e^{\beta\varphi}\, L^i_\alpha\, \nu^\alpha \,, \tag{2.19}$$

where

$$h_{\alpha\beta} = L^i_\alpha L^i_\beta \,. \tag{2.20}$$

Noting that $d\widehat{e}^A = -\widehat{\omega}^A{}_B \wedge \widehat{e}^B$ with $\widehat{\omega}_{AB} = -\widehat{\omega}_{BA}$, one finds [1]

$$\begin{aligned}
\widehat{\omega}_{ab} &= \omega_{ab} + \alpha\, e^{-\alpha\varphi}\, (\partial_b\varphi\, \eta_{ac} - \partial_a\varphi\, \eta_{bc}\,)\, \hat{e}^c + \tfrac{1}{2} e^{(\beta-2\alpha)\,\varphi}\, F^i_{ab}\, \hat{e}^i \ , \\
\widehat{\omega}_{ai} &= -e^{-\alpha\varphi}\, P_{a\,ij}\, \hat{e}^j - \beta\, e^{-\alpha\varphi}\, \partial_a\varphi\, \hat{e}^i + \tfrac{1}{2} e^{(\beta-2\alpha)\,\varphi}\, F^i_{ab}\, \hat{e}^b \ , &(2.21)\\
\widehat{\omega}_{ij} &= e^{-\alpha\varphi}\, Q_{a\,ij}\, \hat{e}^a + \tfrac{1}{2}\, e^{-\beta\varphi}\, C_{k,\,ij}\, \hat{e}^k \ , &(2.22)
\end{aligned}$$

where $F^i = F^\alpha L^i_\alpha$ and

$$C_{k,\,ij} = \Big(L_{\alpha k}\, L^\beta_i\, L^\gamma_j + L_{\alpha j}\, L^\beta_i\, L^\gamma_k - L_{\alpha i}\, L^\beta_j\, L^\gamma_k\Big)\ f^\alpha{}_{\beta\gamma}\,. \tag{2.23}$$

Using these formulae, one finds

$$\begin{aligned}
\int_{M_{10}} \widehat R\ \hat{*}\mathbb{1} &= \frac{1}{n!}\int_G d^n y\, \epsilon_{\alpha_1\cdots\alpha_n} \nu^{\alpha_1}\wedge\cdots\wedge\nu^{\alpha_n}\int_{M_d} \widehat R_{(d)}\, {*\mathbb{1}}\\
&= \frac{1}{n!}\int_G d^n y\, \epsilon_{\alpha_1\cdots\alpha_n} \sigma^{\alpha_1}\wedge\cdots\wedge\sigma^{\alpha_n}\int_{M_d} \widehat R_{(d)}\, {*\mathbb{1}}\\
&= \mathrm{vol}(G)\int_{M_d} \widehat R_{(d)}\, {*\mathbb{1}}\ ,
\end{aligned} \tag{2.24}$$

where $\mathrm{vol}(G)$ is the volume of the group manifold, and

$$\begin{aligned}
\widehat R_{(7)}{*\mathbb{1}} &= R\,{*\mathbb{1}} - \tfrac12 {*d\varphi}\wedge d\varphi - {*P_{ij}}\wedge P_{ij} - \tfrac12 e^{2(\beta-\alpha)\varphi}\, h_{\alpha\beta}\, {*\mathcal{F}^\alpha}\wedge \mathcal{F}^\beta\\
&\quad -\tfrac14\, e^{2(\alpha-\beta)\varphi}\,(h_{\alpha\beta}\, h^{\gamma\delta}\, h^{\rho\sigma}\, f^\alpha{}_{\gamma\rho}\, f^\beta{}_{\delta\sigma} + 2h^{\alpha\beta}\, f^\gamma{}_{\delta\alpha}\, f^\delta{}_{\gamma\beta})\ {*\mathbb{1}}\,.
\end{aligned} \tag{2.25}$$

Here, we have used the notation:

$$P_{a\,ij} \equiv L^\alpha_{(i}\, D_a L_{j)\alpha}\ ,\qquad Q_{a\,ij} \equiv L^\alpha_{[i}\, D_a L_{j]\alpha}\ , \tag{2.26}$$

where L^α_i is the inverse of L^i_α, and our symmetrizations and antisymmetrizations are always with unit strength. It is important to note that no raising or lowering of the group indices has been performed in obtaining (2.25), and so this expression is valid for all the type A groups under consideration, even in cases where the Cartan-Killing metric may be degenerate.

The Yang-Mills Fields:

Next, we consider the ansatz for the Yang-Mills fields:

$$\widehat A^I = A^I + \phi^I_\alpha\, \nu^\alpha\,. \tag{2.27}$$

Defining the lower-dimensional components as

$$\widehat F^I = \big(F^I - \mathcal{F}^\alpha \phi^I_\alpha\big) + P^I_\alpha\wedge\nu^\alpha + \tfrac12 F^I_{\alpha\beta}\,\nu^\alpha\wedge\nu^\beta\ , \tag{2.28}$$

one finds that

$$\begin{aligned}
P^I_\alpha &= D\phi^I_\alpha\ ,\\
F^I_{\alpha\beta} &= f^I{}_{JK}\,\phi^J_\alpha\phi^K_\beta - f^\gamma{}_{\alpha\beta}\,\phi^I_\gamma\ ,
\end{aligned} \tag{2.29}$$

where

$$
\begin{aligned}
F^I &= dA^I + \tfrac{1}{2} f^I{}_{JK} A^J \wedge A^K \ , &(2.30)\\
D\phi^I_\alpha &= d\phi^I_\alpha + f^\gamma{}_{\alpha\beta} \mathcal{A}^\beta \phi^I_\gamma + f^I{}_{JK}\, A^J \phi^K_\alpha \ . &(2.31)
\end{aligned}
$$

As the ten-dimensional theory involves Yang-Mills Chern-Simons forms, it is also useful to consider their dimensional reduction. We define the terms in the lower dimension by setting

$$
\begin{aligned}
\widehat{\omega}_{3Y} &= \widehat{F}^I \wedge \widehat{A}^I - \tfrac{1}{6}\, f_{IJK}\, \widehat{A}^I \wedge \widehat{A}^J \wedge \widehat{A}^K \ ,\\
&= \omega'_{3Y} + \omega_{(2)\alpha}\, \nu^\alpha + \tfrac{1}{2} \omega_{(1)\alpha\beta}\, \nu^\alpha \wedge \nu^\beta + \tfrac{1}{6} \omega_{(0)\alpha\beta\gamma}\, \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma \, . &(2.32)
\end{aligned}
$$

From (2.27) and (2.28) it then follows that

$$
\begin{aligned}
\omega'_{3Y} &= \omega_{3Y} - \phi^I_\alpha F^\alpha \wedge A^I,\\
\omega_{(2)\alpha} &= \Big(F^I - \mathcal{F}^\beta \phi^I_\beta\Big)\, \phi^I_\alpha - P^I_\alpha \wedge A^I - \tfrac{1}{2} f_{IJK} A^I \wedge A^J \phi^K_\alpha \ ,\\
\omega_{(1)\alpha\beta} &= 2 P^I_{[\alpha}\, \phi^I_{\beta]} + F^I_{\alpha\beta} A^I - f_{IJK}\, A^I \phi^J_\alpha \phi^K_\beta \ ,\\
\omega_{(0)\alpha\beta\gamma} &= 3 F^I_{[\alpha\beta} \phi^I_{\gamma]} - f_{IJK} \phi^I_\alpha \phi^J_\beta \phi^K_\gamma \ . &(2.33)
\end{aligned}
$$

The 2-Form Potential:

Next, we consider the 2-form potential, for which the reduction ansatz will be

$$
\widehat{B} = m\, \omega_{(2)} + B + B_\alpha \wedge \nu^\alpha + \tfrac{1}{2} B_{\alpha\beta}\, \nu^\alpha \wedge \nu^\beta \, . \tag{2.34}
$$

Here we have introduced $\omega_{(2)}$, defined by the requirement that

$$
d\omega_{(2)} = \tfrac{1}{6} m f_{\alpha\beta\gamma}\, \sigma^\alpha \wedge \sigma^\beta \wedge \sigma^\gamma \, . \tag{2.35}
$$

The lowering of the index on the structure constants of the group G is performed using the Cartan-Killing metric $\eta_{\alpha\beta}$:

$$
f_{\alpha\beta\gamma} = \eta_{\gamma\delta}\, f^\delta{}_{\alpha\beta} \ , \qquad \eta_{\alpha\beta} = -\tfrac{1}{2}\, f^\gamma{}_{\delta\alpha}\, f^\delta{}_{\gamma\beta} \ . \tag{2.36}
$$

This is well-defined even for non-semisimple Lie algebras where $\eta_{\alpha\beta}$ may be degenerate, or even vanishing. Since we wish to cover these cases as well, the group indices will never be raised with the inverse metric $\eta^{\alpha\beta}$, since this may not exist. We define lower-dimensional field strengths by writing

$$
d\widehat{B} = G^{(0)} + G^{(0)}_\alpha \wedge \nu^\alpha + \tfrac{1}{2} G^{(0)}_{\alpha\beta} \wedge \nu^\alpha \wedge \nu^\beta + \tfrac{1}{6} G^{(0)}_{\alpha\beta\gamma}\, \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma \ . \tag{2.37}
$$

It follows that [40]

$$
\begin{aligned}
G^{(0)} &= dB + B_\alpha \wedge \mathcal{F}^\alpha + \tfrac{1}{6} m\, f_{\alpha\beta\gamma}\, \mathcal{A}^\alpha \wedge \mathcal{A}^\beta \wedge \mathcal{A}^\gamma \\
G^{(0)}_\alpha &= DB_\alpha + B_{\alpha\beta}\, \mathcal{F}^\beta + \tfrac{1}{2} m\, f_{\alpha\beta\gamma}\, \mathcal{A}^\beta \wedge \mathcal{A}^\gamma\ , \\
G^{(0)}_{\alpha\beta} &= DB_{\alpha\beta} + f^\gamma{}_{\alpha\beta}\, B_\gamma + m\, f_{\gamma\alpha\beta}\, \mathcal{A}^\gamma\ , \\
G^{(0)}_{\alpha\beta\gamma} &= -3 B_{\delta[\alpha}\, f^\delta{}_{\beta\gamma]} + m\, f_{\alpha\beta\gamma}\ ,
\end{aligned}
\tag{2.38}
$$

where $DB_\alpha = dB_\alpha + f^\gamma{}_{\alpha\beta} \mathcal{A}^\beta B_\gamma$. Note that for $n = 3$, the first term on the right hand side of $G_{(0)\alpha\beta\gamma}$ vanishes. Expanding $\widehat{G}_{(3)}$,

$$
\widehat{G}_{(3)} = G_{(3)} + G_{(2)\alpha} \wedge \nu^\alpha + \tfrac{1}{2} G_{(1)\alpha\beta} \wedge \nu^\alpha \wedge \nu^\beta + \tfrac{1}{6} G_{(0)\alpha\beta\gamma}\, \nu^\alpha \wedge \nu^\beta \wedge \nu^\gamma\ , \tag{2.39}
$$

where the subscripts denote the form degree, we find

$$
\begin{aligned}
G_{(3)} &= G^{(0)} - \tfrac{1}{2} \omega'_{3Y}\ , \\
G_{(2)\alpha} &= G^{(0)}_\alpha - \tfrac{1}{2} \omega_{(2)\alpha}\ , \\
G_{(1)\alpha\beta} &= G^{(0)}_{\alpha\beta} - \tfrac{1}{2} \omega_{(1)\alpha\beta}\ , \\
G_{\alpha\beta\gamma} &= G^{(0)}_{\alpha\beta\gamma} - \tfrac{1}{2} \omega_{(0)\alpha\beta\gamma}\ ,
\end{aligned}
\tag{2.40}
$$

where the $G^{(0)}$ are given in (2.38) and the Chern-Simons terms in (2.33).

These results suggest that one make the field redefinition

$$
C_\alpha \equiv B_\alpha + \tfrac{1}{2} A^I \phi^I_\alpha\ . \tag{2.41}
$$

This is also motivated, as we shall see below, by the fact that C_α is invariant under the gauge transformations of the Yang-Mills group K, while B_α is not. With this redefinition, the results (2.40) take the form:

$$
G_{(3)} = dB_2 - \tfrac{1}{2} \omega_{3Y} + C_\alpha \wedge \mathcal{F}^\alpha + \tfrac{1}{6} m f_{\alpha\beta\gamma} \mathcal{A}^\alpha \mathcal{A}^\beta \mathcal{A}^\gamma\ , \tag{2.42}
$$

$$
G_{(2)\alpha} = DC_\alpha + C_{\alpha\beta} \mathcal{F}^\beta - F^I \phi^I_\alpha + \tfrac{1}{2} m f_{\alpha\beta\gamma} \mathcal{A}^\beta \wedge \mathcal{A}^\gamma\ , \tag{2.43}
$$

$$
G_{(1)\alpha\beta} = DB_{\alpha\beta} + f^\gamma{}_{\alpha\beta} C_\gamma - P^I_{[\alpha} \phi^I_{\beta]} + m f_{\alpha\beta\gamma} \mathcal{A}^\gamma\ , \tag{2.44}
$$

$$
G_{\alpha\beta\gamma} = 3 f^\delta{}_{[\alpha\beta}\, C_{\gamma]\delta} - f_{IJK} \phi^I_\alpha \phi^J_\beta \phi^K_\gamma + m f_{\alpha\beta\gamma}\ , \tag{2.45}
$$

where we have defined

$$
C_{\alpha\beta} = B_{\alpha\beta} + \tfrac{1}{2} \phi^I_\alpha \phi^I_\beta\ . \tag{2.46}
$$

The resulting Bianchi identities are

$$
dG_{(3)} = G_\alpha \wedge \mathcal{F}^\alpha - \tfrac{1}{2} \left(\mathcal{F}^\alpha \phi^I_\alpha - F^I \right) \Big(\mathcal{F}^\beta \phi^I_\beta - F^I \Big)\ , \tag{2.47}
$$

$$
\begin{aligned}
DG_{(2)\alpha} &= G_{\alpha\beta} \wedge \mathcal{F}^\beta + P^I_\alpha \wedge \Big(\mathcal{F}^\beta \phi^I_\beta - F^I\Big) \ , && (2.48)\\
DG_{(1)\alpha\beta} &= G_{\alpha\beta\gamma}\, \mathcal{F}^\gamma + P^I_\alpha \wedge P^I_\beta \\
&\quad + f^\gamma{}_{\alpha\beta} \left(G_{(2)\gamma} + F^I \phi^I_\gamma\right) + f_{IJK} \phi^I_\alpha \phi^J_\beta \left(\mathcal{F}^\gamma \phi^K_\gamma - F^K\right) \ . && (2.49)
\end{aligned}
$$

Finally, we note that the gauge transformation of the 2-form potential,

$$
\delta \widehat{B} = \tfrac{1}{2} \widehat{A}^I d\widehat{\Lambda}^I \ , \tag{2.50}
$$

implies the following Λ_I gauge transformations in seven dimensions:

$$
\delta B = \tfrac{1}{2} A^I \wedge d\Lambda^I \ , \qquad \delta C_\alpha = 0 \ , \qquad \delta B_{\alpha\beta} = 0 \, . \tag{2.51}
$$

3 The Model in d Dimensions

3.1 The bosonic Lagrangian

The reduction of the kinetic term for the 2-form potential yields the result [1, 40]

$$
\begin{aligned}
\mathcal{L}_3 &= -\tfrac{1}{2} e^{-(\hat\phi + 4\alpha\varphi)} {*G_{(3)}} \wedge G_{(3)} - \tfrac{1}{2} e^{-\hat\phi - 2(\alpha+\beta)\varphi} {*G_{(2)\alpha}} \wedge G^\alpha_{(2)} \\
&\quad -\tfrac{1}{2} e^{-(\hat\phi + 4\beta\varphi)} {*G_{(1)\alpha\beta}} \wedge G^{\alpha\beta}_{(1)} - \tfrac{1}{12} m^2 e^{-\hat\phi + 2(\alpha - 3\beta)\varphi} {*G_{(0)\alpha\beta\gamma}}\, G^{\alpha\beta\gamma}_{(0)} \ , && (3.1)
\end{aligned}
$$

and the reduction of the Yang-Kinetic term gives

$$
\begin{aligned}
\mathcal{L}_2 &= -\tfrac{1}{2} e^{-(\hat\phi + 4\varphi)/2} * \left(F^I - \mathcal{F}^\alpha\, \phi^I_\alpha\right) \wedge \Big(F^I - \mathcal{F}^\beta\, \phi^I_\beta\Big) - \tfrac{1}{2} e^{-(\hat\phi + 4\beta\varphi)/2}\ {*P^I_\beta} \wedge P^{I\beta} \\
&\quad -\tfrac{1}{4} e^{-\frac{1}{2}\hat\phi + 2(\alpha - 2\beta)\varphi}\ {*F^I_{\alpha\beta}}\, F^{I\alpha\beta} \ . && (3.2)
\end{aligned}
$$

It is important to note that any raising of the group manifold indices from their original positions is performed with $h^{\alpha\beta}$, which is the always well-defined inverse of the unimodular scalar matrix $h_{\alpha\beta}$. Thus, for example,

$$
F^{I\alpha\beta} \equiv h^{\alpha\gamma}\, h^{\beta\delta}\, F^I_{\gamma\delta} \, . \tag{3.3}
$$

Both the results given above are up to the group manifold volume factor. In order to identify the factor multiplying the 3-form kinetic term as the dilaton of the reduced theory, and to have canonically-normalized scalar kinetic terms, we need to define the scalar fields in d dimensions as

$$
\phi = a^{-1} (\hat a\, \widehat{\phi} - 4\alpha\, \varphi) \, , \qquad \sigma = a^{-1} (\hat a\, \varphi + 4\alpha\, \widehat{\phi}) \, , \tag{3.4}
$$

with

$$a^2 = \frac{8}{d-2}\,. \tag{3.5}$$

These imply

$$\varphi = a^{-1}(\hat{a}\,\sigma - 4\alpha\,\phi)\,, \qquad \widehat{\phi} = a^{-1}(\hat{a}\,\phi + 4\alpha\,\sigma)\,. \tag{3.6}$$

With these definitions at hand, the sum of $\mathcal{L}_3$, $\mathcal{L}_2$, the Einstein-Hilbert term and the kinetic terms for $\widehat{\phi}$ produces the total d-dimensional bosonic Lagrangian

$$\begin{aligned}
\mathcal{L}_B &= R*\mathbb{1} - \tfrac{1}{2}{*d\phi}\wedge d\phi - \tfrac{1}{2}e^{a\,\phi}\,{*G_{(3)}}\wedge G_{(3)} \\
&\quad -\tfrac{1}{2}{*d\sigma}\wedge d\sigma - {*P^{ij}}\wedge P^{ij} - \tfrac{1}{2}\,{*P^{iI}}\wedge P^{iI} - \tfrac{1}{2}{*G^{ij}_{(1)}}\wedge G^{ij}_{(1)} \\
&\quad -\tfrac{1}{2}e^{a\,\phi/2}\,{*\mathcal{F}^i}\wedge\mathcal{F}^i - \tfrac{1}{2}e^{a\phi/2}\,*G^i_{(2)}\wedge G^i_{(2)} \\
&\quad -\tfrac{1}{2}e^{a\phi/2}\,*\left(F^I - \mathcal{F}^\alpha\,\phi^I_\alpha\right)\wedge\left(F^I - \mathcal{F}^\beta\,\phi^I_\beta\right) - V*\mathbb{1}\,,
\end{aligned} \tag{3.7}$$

where the potential V for the scalar fields is given by

$$\begin{aligned}
V &= \tfrac{1}{4}e^{-a\phi/2}\left(F^I_{ij}F^{Iij} + \tfrac{1}{3}\,G_{ijk}\,G^{ijk}\right) \\
&\quad +\tfrac{1}{4}\,e^{-(a\,\phi+b\,\sigma)/2}\left(h_{\alpha\beta}\,h^{\gamma\delta}\,h^{\rho\sigma}\,f^\alpha{}_{\gamma\rho}\,f^\beta{}_{\delta\sigma} + 2h^{\alpha\beta}\,f^\gamma{}_{\delta\alpha}\,f^\delta{}_{\gamma\beta}\right)\,.
\end{aligned} \tag{3.8}$$

Note that we have made the definitions

$$\begin{aligned}
\mathcal{F}^i &= \mathcal{F}^\alpha\,L^i_\alpha e^{b\sigma/4}\,, & F^I_{ij} &= F^I_{\alpha\beta}\,L^\alpha_i L^\beta_j\,e^{-b\sigma/2}\,, \\
G_{(2)i} &= G_{(2)\alpha}\,L^\alpha_i\,e^{-b\sigma/4}\,, & G_{(1)ij} &= G_{(1)\alpha\beta}\,L^\alpha_i L^\beta_j\,e^{-b\sigma/2} \\
G_{ijk} &= G_{\alpha\beta\gamma}\,L^\alpha_i L^\beta_j L^\gamma_j\,e^{-3b\sigma/4}\,, & P^{iI} &= P_\alpha L^{\alpha i}\,e^{-b\sigma/4}\,,
\end{aligned} \tag{3.9}$$

where

$$b = \sqrt{\frac{8}{n}}\,. \tag{3.10}$$

The raising and lowering of the indices $i, j, .. = 1, ..., n$ and $I = 1, ..., \dim K$ are performed with the Kronecker deltas δ_{ij} and δ_{IJ}.

It is also useful to write the potential as the sum of squares of the functions that appear in the supersymmetry transformation rules. The result is:

$$V = -\tfrac{1}{24}\left(G_{ijk} - 3e^{-b\sigma/4}C_{[i.jk]}\right)^2 + \tfrac{1}{8}\left(G_{ijk} - e^{-b\sigma/4}C_{i.jk}\right)^2 + \tfrac{1}{4}\left(F^I_{ij}\right)^2\,. \tag{3.11}$$

3.2 The supersymmetry transformation rules

The group reduction of supersymmetry transformations were first studied in detail in [41] in the context of $SU(2)$ reduction of the eleven dimensional supergravity. We refer to that work for some details, such as the compensating gauge transformations that play a significant role. Here we shall simply give our results. One improvement is that not only are our results valid for the group reduction of supergravities in arbitrary dimensions, but we also settle fully the problem of diagonalisation of all the fermionic kinetic terms. In [41] and many other works that followed, typically only a partial diagonalisation of the gravitino kinetic term was performed.

In order to obtain the supersymmetry transformation rules for the vielbein and dilaton, and the kinetic terms for all the fermions in canonical diagonalized form, we find that we need to define the d-dimensional fermions and supersymmetry parameters as follows:

$$\begin{aligned}
\psi_a &= e^{\alpha\varphi/2}\left(\widehat{\psi}_a + \tfrac{a^2}{8}\,\Gamma_a\Gamma^i\widehat{\psi}_i\right)\,,\\
\psi_i &= e^{\alpha\varphi/2}\left(\widehat{\psi}_i + \tfrac{\hat{a}}{4}\,\Gamma_i\widehat{\chi}\right)\,,\\
\chi &= e^{\alpha\varphi/2}\left(-\tfrac{\hat{a}}{a}\,\widehat{\chi} + \tfrac{a}{2}\,\Gamma^i\widehat{\psi}_i\right)\,,\\
\lambda &= e^{a\phi/2}\,\widehat{\lambda}\,,\\
\epsilon &= e^{-\alpha\varphi/2}\hat{\epsilon}\,.
\end{aligned} \tag{3.12}$$

The inverse relations are also useful:

$$\begin{aligned}
\hat{\psi}_a &= e^{-\alpha\varphi/2}\left(\psi_a - \tfrac{1}{8}\hat{a}^2\,\Gamma_a\Gamma^i\psi_i - \tfrac{1}{32}na\hat{a}^2\,\Gamma_a\chi\right)\,,\\
\hat{\psi}_i &= e^{-\alpha\varphi/2}\left(\psi_i - \tfrac{1}{8}\hat{a}^2\Gamma_i\Gamma^j\psi_j + \tfrac{\hat{a}^2}{4a}\Gamma_i\chi\right)\,,\\
\hat{\chi} &= e^{-\alpha\varphi/2}\left(-\tfrac{\hat{a}}{a}\chi + \tfrac{\hat{a}}{2}\Gamma^i\psi_i\right)\,.
\end{aligned} \tag{3.13}$$

Note that Γ_μ and Γ^i are both 32×32 and have the symmetry properties inherited from the Γ-matrices in $D=10$. In particular they obey the Clifford algebra $\{\Gamma_\mu,\Gamma_\nu\}=2\eta_{\mu\nu}$, $\{\Gamma_i,\Gamma_j\}=2\delta_{ij}$, and $\{\Gamma_\mu,\Gamma_i\}=0$. It is convenient to work with these matrices to exploit the familiar properties they inherit from $D=10$ inheritance, and also to avoid clutter in notation. If desired, it is an easy matter to express Γ_μ and Γ^i as direct product of suitable Γ matrices of the Clifford algebras in d and n dimensions.

With the above definitions of the fermionic fields in d dimensions, their kinetic Lagrangian is completely diagonalized and takes the form

$$\mathcal{L} = -\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho - \tfrac{1}{2}\bar{\chi}\Gamma^\mu D_\mu\chi - \bar{\psi}_i\Gamma^\mu D_\mu\psi_i - \tfrac{1}{2}\bar{\lambda}^I\Gamma^\mu D_\mu\lambda^I\,. \tag{3.14}$$

Next, we compute the supersymmetry transformation rules. A straightforward calculation yields for the bosons

$$\delta e_\mu^a = \tfrac{1}{2}\bar\epsilon\Gamma^a\psi_\mu\,, \tag{3.15}$$

$$\delta\phi = \tfrac{1}{2}\bar\epsilon\chi\,, \tag{3.16}$$

$$\delta B_{\mu\nu} = e^{-a\phi/2}\left(\bar\epsilon\Gamma_\mu\psi_\nu - \tfrac{a}{4}\,\bar\epsilon\Gamma_{\mu\nu}\chi\right) - 2\delta A^\alpha_{[\mu}\,C_{\nu]\alpha} - A^I_{[\mu}\delta A^I_{\nu]}\,, \tag{3.17}$$

$$L^i_\alpha\delta\mathcal{A}^\alpha_\mu = e^{-(a\phi+b\sigma)/4}\left(-\tfrac{1}{2}\bar\epsilon\Gamma^i\psi_\mu - \tfrac{a}{8}\,\bar\epsilon\Gamma_\mu\Gamma^i\chi - \tfrac{1}{2}\bar\epsilon\Gamma_\mu\psi^i\right)\,, \tag{3.18}$$

$$\begin{aligned} L_i^\alpha\delta C_{\mu\alpha} &= e^{(b\sigma-a\phi)/4}\left(-\tfrac{1}{2}\bar\epsilon\Gamma_i\psi_\mu - \tfrac{a}{8}\,\bar\epsilon\Gamma_\mu\Gamma_i\chi + \tfrac{1}{2}\bar\epsilon\Gamma_\mu\psi_i\right) \\ &\quad + L_i^\alpha\left(-\delta\mathcal{A}^\beta_\mu\,C_{\alpha\beta} + \delta A^I_\mu\phi^I_\alpha\right)\,, \end{aligned} \tag{3.19}$$

$$\delta A^I_\mu = \tfrac{1}{2}e^{-a\phi/4}\,\bar\epsilon\Gamma_\mu\lambda^I + \delta\mathcal{A}^\alpha_\mu\,\phi^I_\alpha\,, \tag{3.20}$$

$$\delta\sigma = -\tfrac{b}{4}\,\bar\epsilon\Gamma^i\psi_i\,,$$

$$L_i^\alpha L_j^\beta\delta B_{\alpha\beta} = e^{b\sigma/2}\,\bar\epsilon\Gamma_{[i}\psi_{j]} - \phi^I_{[i}\delta\phi^I_\beta\,L^\beta_{j]}\,, \tag{3.21}$$

$$L_i^\alpha\delta L_{\alpha j} = \tfrac{1}{2}\bar\epsilon\Gamma_{(i}\psi_{j)} - \tfrac{1}{2n}\delta_{ij}\,\bar\epsilon\Gamma^k\psi_k\,, \tag{3.22}$$

$$L_i^\alpha\delta\phi^I_\alpha = \tfrac{1}{2}e^{b\sigma/4}\,\bar\epsilon\Gamma_i\lambda^I\,, \tag{3.23}$$

where $\phi^I_i = \phi^I_\alpha\,L^\alpha_i$. Our results for the supersymmetry transformations of the fermionic fields are:

$$\begin{aligned} \delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \tfrac{1}{96}\,e^{a\phi/2}\left(a^2\Gamma_\mu\Gamma^{\nu\rho\sigma} - 12\delta^\nu_\mu\Gamma^{\rho\sigma}\right)\,G_{\nu\rho\sigma}\,\epsilon \\ &\quad + \tfrac{1}{64}\,e^{a\phi/4}\left(a^2\Gamma_\mu\Gamma^{\nu\rho} - 16\delta^\nu_\mu\Gamma^\rho\right)(\mathcal{F}_{\mu\nu i} + G_{\mu\nu i})\,\Gamma^i\,\epsilon \\ &\quad - \tfrac{a^2}{192}\,e^{-a\phi/4}\,\left(G_{kij} - 3e^{-b\sigma/4}\,C_{k,\,ij}\right)\,\Gamma_\mu\Gamma^{ijk}\epsilon\,, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \delta\chi &= \tfrac{1}{2}\Gamma^\mu\partial_\mu\phi\,\epsilon + \tfrac{a^2}{24}\,e^{a\phi/2}\,\Gamma^{\mu\nu\rho}G_{\mu\nu\rho}\,\epsilon + \tfrac{a}{16}\,e^{a\phi/4}\,(\mathcal{F}_{\mu\nu i} + G_{\mu\nu i})\,\Gamma^{\mu\nu}\Gamma^i\,\epsilon \\ &\quad - \tfrac{a}{48}\,e^{-a\phi/4}\left(G_{kij} - 3e^{-b\sigma/4}\,C_{k,\,ij}\right)\,\Gamma^{ijk}\,\epsilon\,, \end{aligned} \tag{3.25}$$

$$\begin{aligned} \delta\psi_i &= \tfrac{1}{8}\,b\,\Gamma_i\Gamma^\mu\partial_\mu\sigma\,\epsilon + \tfrac{1}{4}\left(G_{\mu ij} - 2P_{\mu ij}\right)\,\Gamma^\mu\Gamma^j\,\epsilon - \tfrac{1}{8}\,e^{a\phi/4}\,\Gamma^{\mu\nu}\left(\mathcal{F}_{\mu\nu i} - G_{\mu\nu i}\right)\,\epsilon \\ &\quad - \tfrac{1}{8}\,e^{-a\phi/4}\left(G_{ijk} - e^{-b\sigma/4}\,C_{i,jk}\right)\,\Gamma^{jk}\,\epsilon\,, \end{aligned} \tag{3.26}$$

$$\delta\lambda^I = -\tfrac{1}{4}e^{\alpha\phi/4}\,\left[\,\Gamma^{\mu\nu}\left(F^I_{\mu\nu} - \mathcal{F}^\alpha_{\mu\nu}\,\phi^I_\alpha\right) + 2\Gamma^\mu\Gamma^i P^{iI}_\mu + \Gamma^{ij}F^I_{ij}\,\right]\epsilon\,, \tag{3.27}$$

where we have used the value $\hat{a}=-1$, and the modified covariant derivative is given by

$$\mathcal{D}_\mu\epsilon = \partial_\mu\epsilon + \tfrac{1}{4}\omega_\mu{}^{ab}\,\Gamma_{ab}\,\epsilon + \tfrac{1}{4}\left(Q_{\mu ij} - \tfrac{1}{2}G_{\mu ij}\right)\,\Gamma^{ij}\epsilon\ . \tag{3.28}$$

In particular, the minimal coupling of gauge fields in this covariant derivative is given by

$$\mathcal{D}\epsilon = \partial_\mu\epsilon + \tfrac{1}{4}\,f^\gamma_{\alpha\beta}\,L^\alpha_i\left[\left(g\,L_{\gamma j} + mL^\delta_j\,\eta_{\gamma\delta}\right)A^\beta_\mu - g\,L^\beta_j B_{\mu\gamma}\right]\Gamma^{ij}\,\epsilon + \cdots \tag{3.29}$$

Note also that the sum $(\mathcal{F}+G_{(2)})$ arises in the transformation rules of the graviton multiplet, and as such these are the graviphotons, while the combination $(\mathcal{F}-G_{(2)})$ occurs in the matter sector.

4 The Hidden Duality Symmetries

4.1 The Bosonic Lagrangian

The group reduction described above gives rise to half-maximal (i.e. 16 supersymmetries) gauged supergravities coupled to n vector multiplets in $d=(10-\dim \mathrm{G})$ dimensions, where $n=\dim$ G. The scalar fields described by the internal metric $h_{\alpha\beta}$ parametrize the coset $SL(n,R)/SO(n)$. The global $SL(n,R)$ symmetry is manifest in the absence of gauging, but it breaks down to $G\subset SL(n,R)$ upon gauging. Taking into account the scalars $(B_{\alpha\beta},\sigma,\phi^I_\alpha)$ as well, one expects that together with $h_{\alpha\beta}$ they should parameterize the enlarged coset $SO(n,n+\dim\mathrm{K})/SO(n)\times SO(n+\dim\mathrm{K})$. Indeed, setting the gauge coupling constants to zero, the kinetic terms for the vector fields take the form [19]

$$\mathcal{L}_{vec} = -\tfrac{1}{2}e^{-a\phi/2}\,{*}(\mathcal{V}^T\mathcal{G})^T\wedge\mathcal{V}^T\mathcal{G}\,, \tag{4.1}$$

where

$$\mathcal{G} = \begin{pmatrix} d\mathcal{A}^\alpha \\ dB_\alpha \\ dA^I \end{pmatrix}\,, \qquad \mathcal{V}\mathcal{V}^T = M\,. \tag{4.2}$$

The matrix M is given by

$$M = \left(\begin{array}{c|c|c} G+\phi^T\phi+C^TG^{-1}C & C^TG^{-1} & -(1+C^TG^{-1})\phi^T \\ \hline G^{-1}C & G^{-1} & -G^{-1}\phi^T \\ \hline -\phi(1+G^{-1}C) & -\phi G^{-1} & \mathbb{1}+\phi G^{-1}\phi^T \end{array}\right) \tag{4.3}$$

where $G_{\alpha\beta}=\hat{L}^i_\alpha\hat{L}^i_\beta$ with the definition $\widehat{L}^i_\alpha = e^{b\sigma/4}L^i_\alpha$, and

$$\mathcal{V} = \left(\begin{array}{c|c|c} \hat{L} & C^T \hat{L}^{T-1} & -\phi^T \\ \hline 0 & \hat{L}^{T-1} & 0 \\ \hline 0 & -\phi \hat{L}^{T-1} & \mathbb{1} \end{array}\right) . \tag{4.4}$$

The inverse of this coset representative is given by

$$\mathcal{V}^{-1} = \left(\begin{array}{c|c|c} \hat{L}^{-1} & \hat{L}^{-1} C & \hat{L}^{-1} \phi^T \\ \hline 0 & \hat{L}^{T} & 0 \\ \hline 0 & \phi & \mathbb{1} \end{array}\right) . \tag{4.5}$$

The matrix M leaves invariant the metric

$$\eta = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.6}$$

i.e. $M \eta M^T = \eta$, and therefore it forms the fundamental representation of $SO(n, n + \dim \mathrm{K})$. It also follows that $\mathcal{V}^{-1} = \eta \mathcal{V}^T \eta$. The Lagrangian (4.1) is clearly invariant under a global $SO(n, n + \dim \mathrm{K})$. Turning on the gauge couplings break this symmetry symmetry down to $K \times G \ltimes R^n$.

The bosonic Lagrangian for the scalars $(\sigma, L^i_\alpha, B_{\alpha\beta})$ can be written as a sigma model on the coset $SO(n, n + \dim \mathrm{K})/SO(n) \times SO(\dim \mathrm{K})$. To this end we define the symmetric Maurer-Cartan form,

$$\mathcal{P} = \tfrac{1}{2}\left(\mathcal{V}^{-1} d\mathcal{V} + (\mathcal{V}^{-1} d\mathcal{V})^T\right) = \left(\begin{array}{c|c|c} P^{ij}_\mu + \frac{b}{4}\partial_\mu \sigma \delta^{ij} & -\frac{1}{2} G_{\mu ij} & -\frac{1}{2} P^{iI}_\mu \\ \hline -\frac{1}{2} G_{\mu ij} & -P^{ij}_\mu - \frac{b}{4}\partial_\mu \sigma \delta^{ij} & -\frac{1}{2} P^{iI}_\mu \\ \hline -\frac{1}{2} P^{iI}_\mu & -\frac{1}{2} P^{iI}_\mu & 0 \end{array}\right) . \tag{4.7}$$

With the aid of this formula, and recalling (4.1), the full bosonic Lagrangian for abelian gauge fields with no gauging can be put into the remarkably simple form

$$\mathcal{L}_B = R * \mathbb{1} - \tfrac{1}{2} {*d\phi} \wedge d\phi - \tfrac{1}{2} e^{a\,\phi} * G_{(3)} \wedge G_{(3)} - \tfrac{1}{2} \mathrm{tr} {*\mathcal{P}} \wedge \mathcal{P} - \tfrac{1}{2} e^{-a\phi/2} * (\mathcal{V}^T \mathcal{G})^T \wedge \mathcal{V}^T \mathcal{G} \, , \tag{4.8}$$

where

$$G_{(3)} = dB_2 - \tfrac{1}{2} \mathcal{G}^{\underline{\alpha}} \wedge \mathcal{B}_{\underline{\alpha}} \, , \tag{4.9}$$

with the underlined Greek indices understood to be contracted by the Cartan-Killing metric given in (4.6).

4.2 The Supersymmetry Transformation Rules

The $SO(n, n+\dim K)$ symmetry can be made manifest in the supersymmetry transformation rules as well. To begin with, we combine the vector fields and fermionic fields as

$$\mathcal{B}_\mu^{\underline{\alpha}} = \begin{pmatrix} \mathcal{A}_\mu^\alpha \\ B_{\mu\alpha} \\ A_\mu^I \end{pmatrix} , \qquad \psi^r = \begin{pmatrix} \psi^i \\ \frac{1}{\sqrt{2}}\lambda^I \end{pmatrix} \tag{4.10}$$

Furthermore, we define

$$\mathcal{V} = \left(\mathcal{V}_{\underline{\alpha}}^i, \mathcal{V}_{\underline{\alpha}}^r\right) , \qquad i = 1, ..., n, \quad r = 1, ..., (n+\dim K) , \tag{4.11}$$

where

$$\mathcal{V}_{\underline{\alpha}}^i = \mathcal{V}_{\underline{\alpha}}^{+i} , \qquad \mathcal{V}_{\underline{\alpha}}^r = \left(\mathcal{V}_{\underline{\alpha}}^{-i}, \mathcal{V}_{\underline{\alpha}}^I\right) , \qquad \mathcal{V}_{\underline{\alpha}}^{\pm i} = \frac{1}{\sqrt{2}}\left(\pm\mathcal{V}_{\underline{\alpha}}^{1i} + \mathcal{V}_{\underline{\alpha}}^{2i}\right) . \tag{4.12}$$

Here, $\mathcal{V}_{\underline{\alpha}}^{1i}$ and $\mathcal{V}_{\underline{\alpha}}^{2i}$ represent the two n by $(2n+\dim K)$ blocks in the matrix $\mathcal{V}$ given in (4.4). Explicitly, we have

$$(\mathcal{V}_{\underline{\alpha}}^i, \mathcal{V}_{\underline{\alpha}}^r) = \left(\begin{array}{c|c|c} \frac{1}{\sqrt{2}}(\hat{L} + C^T\hat{L}^{T-1}) & \frac{1}{\sqrt{2}}(-\hat{L} + C^T\hat{L}^{T-1}) & -\phi^T \\ \hline \frac{1}{\sqrt{2}}\hat{L}^{T-1} & \frac{1}{\sqrt{2}}\hat{L}^{T-1} & 0 \\ \hline -\frac{1}{\sqrt{2}}\phi\hat{L}^{T-1} & -\frac{1}{\sqrt{2}}\phi\hat{L}^{T-1} & \mathbb{1} \end{array}\right) , \tag{4.13}$$

which has the inverse

$$(\mathcal{V}_i^{\underline{\alpha}}, \mathcal{V}_r^{\underline{\alpha}}) = \left(\begin{array}{c|c|c} \frac{1}{\sqrt{2}}\hat{L}^{-1} & \frac{1}{\sqrt{2}}(\hat{L}^{-1}C + \hat{L}^T) & \frac{1}{\sqrt{2}}\hat{L}^{-1}\phi^T \\ \hline -\frac{1}{\sqrt{2}}\hat{L}^{-1} & \frac{1}{\sqrt{2}}(-\hat{L}^{-1}C + \hat{L}^T) & -\frac{1}{\sqrt{2}}\hat{L}^{-1}\phi^T \\ \hline 0 & \phi & \mathbb{1} \end{array}\right) , \tag{4.14}$$

These definitions are needed in order to ensure that the combinations $(\mathcal{A}_\mu + B_\mu)$, representing the graviphotons, and $(\mathcal{A}_\mu - B_\mu)$, which are the external vector fields, transform appropriately in relation to the fermions of the supergravity and vector multiplets respectively. Denoting the coset representative defined above by $\mathcal{V}_{\rm diag}$ and its inverse by $\mathcal{V}_{\rm diag}^{-1}$, they satisfy the relation $\mathcal{V}_{\rm diag}^{-1} = \eta_{\rm diag}\,\mathcal{V}_{\rm diag}^T\,\eta$ where

$$\eta_{\rm diag} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{4.15}$$

Armed with these definitions, we can now show that the supersymmetry transformations of all the bosonic fields simplify dramatically, and they can be written in a manifestly

$SO(n, n+\dim \mathrm{K})$ covariant way, as follows:

$$\delta e^a_\mu = \tfrac{1}{2}\bar{\epsilon}\Gamma^a\psi_\mu \,, \tag{4.16}$$

$$\delta\phi = \tfrac{1}{2}\bar{\epsilon}\chi \,, \tag{4.17}$$

$$\delta B_{\mu\nu} = e^{-a\phi/2}(\bar{\epsilon}\Gamma_{[\mu}\psi_{\nu]} - \tfrac{a}{4}\,\bar{\epsilon}\Gamma_{\mu\nu}\chi) - \delta\mathcal{B}^{\underline{\alpha}}_{[\mu}\,\mathcal{B}_{\nu]\underline{\alpha}} \,, \tag{4.18}$$

$$\delta\mathcal{B}^{\underline{\alpha}}_\mu = -\tfrac{1}{\sqrt{2}}e^{-a\phi/4}(\bar{\epsilon}\Gamma^i\psi_\mu + \tfrac{a}{4}\,\bar{\epsilon}\Gamma_\mu\Gamma^i\chi)\,\mathcal{V}_i^{\underline{\alpha}} + \tfrac{1}{\sqrt{2}}e^{-a\phi/4}\,\bar{\epsilon}\psi^r\,\mathcal{V}_r^{\underline{\alpha}} , \tag{4.19}$$

$$\left(\mathcal{V}^{-1}\delta\mathcal{V}\right)_{ir} = -\tfrac{1}{4}\bar{\epsilon}\Gamma_i\psi_r \;. \tag{4.20}$$

In the last equation, bearing in mind (4.7) and (4.12), it is understood that $\left(\mathcal{V}^{-1}\delta\mathcal{V}\right)_{ij} = -\frac{1}{2}\left((L^{-1}\delta L)_{ij} + \frac{b}{4}\delta\sigma\delta^{ij}\right) - \frac{1}{4}(L^{-1}\delta B L^{-1})_{ij}$ and that $\left(\mathcal{V}^{-1}\delta\mathcal{V}\right)_{iI} = -\frac{1}{2}(L^{-1}\delta\phi^T)_{iI}$.

The supersymmetry transformations of the fermionic fields also simplify considerably in a manifestly duality-symmetric form as follows:

$$\begin{aligned}\delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \tfrac{1}{96}\,e^{a\phi/2}\left(a^2\Gamma_\mu\Gamma^{\nu\rho\sigma} - 12\delta^\nu_\mu\Gamma^{\rho\sigma}\right)\,G_{\nu\rho\sigma}\,\epsilon \\ &\quad + \tfrac{1}{32}\,e^{a\phi/4}\left(a^2\Gamma_\mu\Gamma^{\nu\rho} - 16\delta^\nu_\mu\Gamma^\rho\right)\mathcal{G}^i_{\mu\nu}\Gamma_i\,\epsilon \,,\end{aligned} \tag{4.21}$$

$$\delta\chi = \tfrac{1}{2}\Gamma^\mu\partial_\mu\phi\,\epsilon + \tfrac{a^2}{24}\,e^{a\phi/2}\,\Gamma^{\mu\nu\rho}G_{\mu\nu\rho}\,\epsilon + \tfrac{a}{8}\,e^{a\phi/4}\mathcal{G}^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma^i\,\epsilon \,, \tag{4.22}$$

$$\delta\psi^r = \mathcal{P}^{ir}_\mu\Gamma^\mu\Gamma^i\epsilon - \tfrac{1}{4}e^{a\phi/4}\mathcal{G}^r_{\mu\nu}\Gamma^{\mu\nu}\epsilon \,, \tag{4.23}$$

where $\mathcal{G}^i = \mathcal{G}^{\underline{\alpha}}\,\mathcal{V}^i_{\underline{\alpha}}$ and $\mathcal{G}^r = \mathcal{G}^{\underline{\alpha}}\,\mathcal{V}^r_{\underline{\alpha}}$. The covariant derivative of the supersymmetry parameter reads

$$\mathcal{D}_\mu\epsilon = \partial_\mu\epsilon + \tfrac{1}{4}\omega_\mu{}^{ab}\,\Gamma_{ab}\,\epsilon + \tfrac{1}{4}\mathcal{Q}_{\mu ij}\,\Gamma^{ij}\epsilon \;, \tag{4.24}$$

with the $SO(n)$-valued composite connection given by $\mathcal{Q}_{\mu ij} = \mathcal{V}^{-1}_{[i}\partial_\mu\mathcal{V}_{j]}$. Here we have used the definitions given in (4.12).

4.3 The nonabelian model

The hidden $SO(n, n+\dim K)/SO(n)\times SO(n+\dim K)$ coset structure can be maintained in the gauged model with nonabelian gauge symmetry as well. To show this, we begin with the observation that the nonabelian field strength

$$\mathcal{G}^{\underline{\alpha}} = d\mathcal{B}^\alpha + \tfrac{1}{2}\hat{f}^{\underline{\alpha}}{}_{\underline{\beta\gamma}}\,\mathcal{B}^{\underline{\beta}}\wedge\mathcal{B}^{\underline{\gamma}} \,, \qquad \underline{\alpha} = \alpha, \alpha', I \;, \tag{4.25}$$

produces the three field strengths (2.17), (2.30) and (2.43) with the only nonvanishing structure constants given by

$$\hat{f}^\alpha{}_{\beta\gamma} = f^\alpha{}_{\beta\gamma} \;, \quad \hat{f}^{\alpha'}{}_{\beta\gamma} = m\,f_{\alpha\beta\gamma} \;, \quad \hat{f}^{\alpha'}{}_{\beta\gamma'} = -\hat{f}^{\alpha'}{}_{\gamma'\beta} = -f^\gamma{}_{\beta\alpha} \;, \quad \hat{f}^I{}_{JK} = f^I{}_{JK} \;, \tag{4.26}$$

These define the semi-direct-product group $K \times G \ltimes R^n$, and we have the associated field strengths

$$\mathcal{G}^{\underline{\alpha}}\mathcal{V}_{\underline{\alpha}} = \begin{pmatrix} \mathcal{F}^{\alpha} \\ G_{(2)\alpha} \\ F^I - \mathcal{F}^{\underline{\alpha}}\phi^I_\alpha \end{pmatrix} \,. \tag{4.27}$$

With these identifications, we find that the field strength $G_{(3)}$ defined in (2.42) can also be written in a manifestly $K \times G \ltimes R^n$ invariant form as

$$G_{(3)} = dB_2 - \tfrac{1}{2}\Big(\mathcal{G}^{\underline{\alpha}} \wedge \mathcal{B}_{\underline{\alpha}} - \tfrac{1}{6}\hat{f}^{\underline{\alpha}}{}_{\underline{\beta\gamma}}\,\mathcal{B}^{\underline{\beta}} \wedge \mathcal{B}^{\underline{\gamma}} \wedge \mathcal{B}_{\underline{\alpha}}\Big) \,, \tag{4.28}$$

where $\mathcal{B}_{\underline{\alpha}} = \mathcal{B}^{\underline{\beta}}\eta_{\underline{\alpha\beta}}$. Next, we turn to the supersymmetry transformation rules.

The supersymmetry transformations of the bosonic fields will be as given in (4.16) - (4.20), while those of the fermionic fields will now take the form

$$\begin{aligned}
\delta\psi_\mu &= \mathcal{D}_\mu\epsilon + \tfrac{1}{96}\,e^{a\phi/2}\left(a^2\Gamma_\mu\Gamma^{\nu\rho\sigma} - 12\delta^\nu_\mu\Gamma^{\rho\sigma}\right) G_{\nu\rho\sigma}\,\epsilon \\
&\quad + \tfrac{1}{32}\,e^{a\phi/4}\left(a^2\Gamma_\mu\Gamma^{\nu\rho} - 16\delta^\nu_\mu\Gamma^\rho\right)\mathcal{G}^i_{\mu\nu}\Gamma_i\,\epsilon - \tfrac{a^2}{48\sqrt{2}}\Gamma_\mu T_{ijk}\Gamma^{ijk}\epsilon\,, \qquad (4.29)\\
\delta\chi &= \tfrac{1}{2}\Gamma^\mu\partial_\mu\phi\,\epsilon + \tfrac{a^2}{24}\,e^{a\phi/2}\,\Gamma^{\mu\nu\rho}G_{\mu\nu\rho}\,\epsilon + \tfrac{a}{8}\,e^{a\phi/4}\mathcal{G}^i_{\mu\nu}\Gamma^{\mu\nu}\Gamma^i\,\epsilon - \tfrac{a}{12\sqrt{2}}T_{ijk}\Gamma^{ijk}\epsilon\,, \qquad (4.30)\\
\delta\psi^r &= \mathcal{P}^{ir}_\mu\Gamma^\mu\Gamma^i\epsilon - \tfrac{1}{4}e^{a\phi/4}\mathcal{G}^r_{\mu\nu}\Gamma^{\mu\nu}\epsilon - \tfrac{1}{2\sqrt{2}}T^r_{ij}\Gamma^{ij}\epsilon\,, \qquad (4.31)
\end{aligned}$$

where the covariant derivative $D_\mu\epsilon$ is defined as in (4.24), with the composite connection now given by $\mathcal{Q}_{ij} = \mathcal{V}^{-1}_{[i}D_\mu\mathcal{V}_{j]}$, involving the $K \times G \ltimes R^n$ covariant derivative.

Finally, we find that the bosonic action can also be written in a manifestly $K \times G \ltimes R^n$ invariant form as

$$\begin{aligned}
\mathcal{L}_B &= R\,{*\mathbb{1}} - \tfrac{1}{2}{*d\phi} \wedge d\phi - \tfrac{1}{2}e^{a\,\phi}\,{*G_{(3)}} \wedge G_{(3)} - \tfrac{1}{2}\mathrm{tr}{*\mathcal{P}} \wedge \mathcal{P} \\
&\quad - \tfrac{1}{2}e^{-a\phi/2}\,{*(\mathcal{V}^T\mathcal{G})^T} \wedge \mathcal{V}^T\mathcal{G} - \tfrac{1}{3}T_{ijk}T^{ijk} + T_{ijr}T^{ijr}\,, \qquad (4.32)
\end{aligned}$$

where

$$\mathcal{P}_\mu = \mathcal{V}^{\underline{\beta}}\left(\partial_\mu\delta^{\underline{\alpha}}_{\underline{\beta}} + \hat{f}^{\underline{\alpha}}{}_{\underline{\beta\gamma}}\,\mathcal{B}^{\underline{\gamma}}_\mu\right)\mathcal{V}_{\underline{\alpha}}\,, \tag{4.33}$$

and the T-tensors are just the boosted structure constants, defined by

$$T_{ijk} = \hat{f}^{\underline{\alpha}}{}_{\underline{\beta\gamma}}\,\mathcal{V}_{\underline{\alpha}i}\,\mathcal{V}^{\underline{\beta}}_j\mathcal{V}^{\underline{\gamma}}_k\,, \qquad T_{ijr} = \hat{f}^{\underline{\alpha}}{}_{\underline{\beta\gamma}}\,\mathcal{V}_{\underline{\alpha}r}\,\mathcal{V}^{\underline{\beta}}_j\mathcal{V}^{\underline{\gamma}}_k\,. \tag{4.34}$$

The sigma model kinetic term involves $\mathcal{P}_\mu$, which takes the same form as in (4.7) but with covariantized field strengths.

5 The Structure and Consistent Truncations

The results we obtained above have the same overall structure as in the couplings of half-maximal d-dimensional supergravities coupled to $(n + \dim K)$ vector multiplets. However, while in a typical gauged supergravity theory it is usually assumed that the gauge group is semisimple, here we have landed on a particular class of theories in which the gauge group is $K \times G \ltimes R^n$, where K is semisimple but G is a completely arbitrary group of dimension $n < 10$ with traceless structure constants. The semi-direct structure brings in restrictions on the consistent truncations of the vector multiplets, as we shall see in the next section. In particular, looking at a case of particular interest, namely the $d = 7$ theory with gauge symmetry $SO(2,1) \ltimes R^3$ (with the gauge fields of group K set to zero), we find that the two vector fields that need to be truncated away in order to obtain the standard $SO(2,1)$ gauged model of [14] do not actually truncate consistently. This is due to the fact that the six vector fields in the theory split into a triplet in the adjoint representation of $SO(2,1)$, with the remaining (external) triplet of vector fields transforming nontrivially under $SO(2,1)$.

On the other hand, a consistent truncation of the external $n + \dim K)$ vector multiplets in which n vector multiplets that gauge the compact R-symmetry group always exists, as we shall see below. We shall first describe this successful truncation, and after that, show explicitly the obstacles one faces in attempting to truncate the theory in $d = 6$ to a gauged chiral supergravity. This is closely related to the consistent truncation problem mentioned above. Indeed, had the standard $SO(2,1)$ gauged theory in $d = 7$ resulted from our model, it is known [14] that it could then by reduction and truncation produce a chiral gauged supergravity in six dimensions.

5.1 The truncation of the external vector multiplets

In d dimensions, the model we have obtained describes the gauged coupling of half-maximal supergravity,

$$(g_{\mu\nu},\ B_{\mu\nu},\ \phi,\ \mathcal{A}^i_\mu + B^i_\mu,\ \psi_\mu,\ \chi)\ , \tag{5.1}$$

with $8(d-2)_B + 8(d-2)_F$ degrees of freedom, coupled to $n + \dim \mathrm{G}$ vector multiplets with field content

$$(\mathcal{A}^i_\mu - B^i_\mu,\ h_{\alpha\beta},\ B_{\alpha\beta},\ \sigma,\ \psi_i)\ , \qquad (A^I_\mu, \lambda^I, \phi^I_\alpha)\,, \tag{5.2}$$

and with $8(n + \dim \mathrm{G})_B + 8(n + \dim \mathrm{G})_F$ degrees of freedom. All the vector multiplets can be consistently decoupled, to leave a gauged half-maximal supergravity with gauge group

G. To see this, we set

$$\begin{aligned} h_{\alpha\beta} &= h^{(0)}_{\alpha\beta}\,, \qquad \mathcal{A}^i_\mu - B^i_\mu = 0\,, \qquad A^I_\mu = 0\,, \\ \sigma &= 0\,, \qquad B_{\alpha\beta} = 0\,, \qquad \phi^I_\alpha = 0\,, \\ \psi_i &= 0\,, \qquad \lambda^I = 0\,, \end{aligned} \tag{5.3}$$

where $h^{(0)}_{\alpha\beta}$ is a constant unimodular 4×4 matrix. With this ansatz, the supersymmetry transformation rules consistently truncate, provided that

$$\mathcal{F}^i_{\mu\nu} - G^i_{\mu\nu} = 0\,, \quad G_{ijk} - C_{k,\,ij} = 0\,, \quad G_{\mu ij} = 0\,, \quad P_{\mu ij} = 0\,, \; \sigma = 0\,. \tag{5.4}$$

The first and second conditions independently give the relation

$$m\,\eta_{\gamma\delta}\, f^\delta{}_{\alpha\beta} + g\left(2f^\delta{}_{\gamma[\alpha}\, h^{(0)}_{\beta]\delta} - f^\delta{}_{\alpha\beta}\, h^{(0)}_{\gamma\delta}\right) = 0\,, \tag{5.5}$$

while the third and fourth conditions, respectively, give the relations

$$f^\delta{}_{\alpha\beta}\left(g\, h^{(0)}_{\gamma\delta} + m\eta_{\gamma\delta}\right) = 0\,, \qquad g\, f^\delta{}_{\gamma(\alpha}\, h^{(0)}_{\beta)\delta} = 0\,. \tag{5.6}$$

All of the above conditions are satisfied by taking

$$h^{(0)}_{\alpha\beta} = \eta_{\alpha\beta} = \delta_{\alpha\beta}\,, \qquad g = -m\,. \tag{5.7}$$

Finally, it is important to ensure that all field equations for the fields that we have eliminated are also satisfied, since not all of them may follow from the integrability conditions of the local supersymmetry transformations. We have checked that indeed all the required field equations are satisfied by the solution (5.7). Thus, the resulting supergravity is indeed gauged half-maximal supergravity with $8(d-2)_B + 8(d-2)_F$ degrees of freedom, with a type A gauge group G associated with the gauge fields $(A^i_\mu + B^i_\mu)$.

5.2 The nonexistence of chiral truncation in $d = 6$

At first sight, the most general chiral truncation would require setting

$$\psi_{\mu-} = 0\;, \quad \chi_+ = 0\;, \tag{5.8}$$

leaving us with the $N = (1,0)$ supergravity and tensor multiplet, which contains the fields $(g_{\mu\nu},\ B_{\mu\nu},\ \phi,\ \psi_{\mu+},\ \chi_-)$, the vector multiplets $(\mathcal{A}^i_\mu - B^i_\mu,\ \psi_{i+})$ and $(A^I_\mu,\ \lambda^I_+)$, and the hypermultiplets $(h_{\alpha\beta},\ B_{\alpha\beta},\ \sigma,\ \psi_{i-})$ and $(\phi^I_\alpha,\ \lambda^I_-)$. However, this approach immediately runs into trouble, as can be seen by examining the $\delta\psi_{\mu-} = 0$ condition, since this gives

$$\mathcal{F}^i_{\mu\nu} + G^i_{\mu\nu} = 0\,, \qquad G_{ijk} - 3e^{-b\sigma/4}\, C_{[k,\,ij]} = 0\,. \tag{5.9}$$

These equations place constraints on the fields of the vector and hypermultiplets listed above. The second condition, for example, imposes a nonlinear algebraic constraint on the scalar fields $(\sigma, h_{\alpha\beta}, \phi^I_\alpha)$, and it is not clear at all if this can be satisfied with any surviving hypermultiplet. This condition alone motivates us to eliminate all the hypermultiplets. Thus, in particular setting $\sigma = 0$, $h_{\alpha\beta} = h^{(0)}_{\alpha\beta}$ with constant $h^{(0)}_{\alpha\beta}$, and $\psi_{i-} = 0$, we find from the requirement $\delta\psi_{i-} = 0$ that

$$P_{\mu ij} = 0 \ , \qquad G_{\mu ij} = 0 \,. \tag{5.10}$$

These constraints, and the second one in (5.9) respectively, give the conditions

$$g\, f^\delta{}_{\gamma(\alpha}\, h^{(0)}_{\beta)\delta} = 0 \,, \tag{5.11}$$

$$f^\delta{}_{\alpha\beta} \left(g\, h^{(0)}_{\gamma\delta} + m\eta_{\gamma\delta} \right) = 0 \,, \tag{5.12}$$

$$m\, \eta_{\alpha\delta}\, f^\delta{}_{\alpha\beta} - 3g\, f^\delta{}_{[\alpha\beta}\, h^{(0)}_{\gamma]\delta} = 0 \,. \tag{5.13}$$

Recalling the requirement of R-symmetry gauging, by which the minimal coupling in (3.29) must be nonvanishing, we find that the first equation above can only be satisfied for $SU(2) \times U(1)$, in which case the remaining two equations clearly cannot be satisfied. Thus, we conclude that the model obtained by group manifold reduction cannot be consistently truncated to yield a gauged chiral $N = (1, 0)$ supergravity in $d = 6$.

6 Conclusions

The heterotic supergravity with gauge symmetry K reduced on a group manifold G of dimension n gives rise to a large class of gauge supergravities in $d = (10 - n)$ dimensions that describe the coupling of $(n + \dim K)$ vector multiplets. The reduction involves a nonvanishing 3-form flux, and the Lie algebra of G must have traceless structure constants in order to ensure the consistency of the reduction at the level of the action. The d-dimensional theory has $K \times G \times R^n$ gauge symmetry, and its couplings are governed by the coset $SO(n, n + K)/SO(n) \times SO(n + K)$, which is parametrized by all the scalars of the theory save the dilaton. The group $SO(n, n+ K)$ is a global duality symmetry group that is broken down to $SO(n) \times SO(n+ K)$ in the presence of gauge couplings. A linear combination of the n Kaluza-Klein vectors and n "winding" vectors coming from the 2-form potential assembles into the adjoint representation of the group G, while the orthogonal combination transforms as matter vectors under this group. These matter vectors and the gauge field of group G can be consistently truncated in any dimension, to yield pure half-maximal gauged supergravity with a compact R-symmetry group. Truncations to yield noncompact gaugings do not seem to be possible.

These results primarily provide an embedding of a large class of half-maximal gauged supergravities in heterotic supergravity, and hence in string theory. They also highlight the absence of such origins for some noncompact gauged supergravities that are of considerable interest in fewer than ten dimensions.

In this paper we have restricted our attention to Lie algebras with traceless structure constants, since these give consistent reductions not only at the level of the field equations, but also at the level of the action. The traceless algebras are referred to as type A Lie algebras. Reduction on group manifolds based on type B algebras, for which the structure constants have non-vanishing traces $f^{\alpha}{}_{\alpha\beta} \neq 0$, is consistent only at the level of field equations. Furthermore, the resulting d-dimensional equations are not derivable from an action. It would be useful to perform the reduction for type B algebras as well, which is a straightforward matter using the techniques of this paper, in order to see whether our conclusions about the further consistent truncations persist. It would also be worthwhile to explore the consequences of the free parameters that arise in a number of Lie algebras occurring in the group reduction.

Acknowledgments

We thank Eric Bergshoeff, Murat Günaydin and Kelly Stelle for useful discussions. The work of H. Lü and C.N.P. is supported in part by DOE grant DE-FG03-95ER40917, and that of E.S. is supported in part by NSF Grant PHY-0555575.

A Lie algebras of dimension 2, 3 and 4

Name	Commutator	Type	$\eta_{\mu\nu}$
$A_{2,1}$	$e_{12} = e_2$	B	$(-\frac{1}{2}, 0)$
$A_{3,1}$	$e_{23} = e_1$	A	$(0, 0, -1)$
$A_{3,2}$	$e_{13} = e_1, e_{23} = e_1 + e_2$	B	$(0, 0, -1)$
$A_{3,3}$	$e_{13} = e_1, e_{23} = e_2$	B	$(0, 0, -1)$
$A_{3,4}$	$e_{13} = e_1, e_{23} = -e_2$	A	$(0, 0, -1)$
$A^a_{3,5}$	$e_{13} = e_1, e_{23} = ae_2\,, (0 < \|a\| < 1)$	B	$(0, 0, -\frac{1}{2}(1 + a^2)$
$A_{3,6}$	$e_{13} = -e_2, e_{23} = e_1$	A	$(0, 0, 1)$
$A^a_{3,7}$	$e_{13} = ae_1 - e_2, e_{23} = e_1 + ae_2\,, (a > 0)$	B	$(0, 0, 1 - a^2)$
$A_{3,8}$	$e_{13} = -2e_2, e_{12} = e_1, e_{23} = e_3$	A	$\{-2, -1, -2\}$
$A_{3,9}$	$e_{13} = e_3, e_{23} = e_1, e_{31} = e_2$	A	$(1, 1, 1)$
$A_{4,1}$	$e_{24} = e_1, e_{34} = e_2$	A	$(0, 0, 0, 0)$
$A^a_{4,2}$	$e_{24} = ae_1, e_{24} = e_2, e_{34} = e_2 + e_3, (a \neq 0)$	A$(a = -2)$	$(0, 0, 0, -\frac{2-a^2}{2})$
$A_{4,3}$	$e_{14} = e_1, e_{34} = e_2$	B	$(0, 0, 0, -\frac{1}{2})$
$A_{4,4}$	$e_{14} = e_1, e_{24} = e_1 + e_2, e_{34} = e_2 + e_3$	B	$(0, 0, 0, -\frac{3}{2})$
$A^{ab}_{4,5}$	$e_{14} = e_1, e_{24} = ae_2, e_{34} = be_3$ $(ab \neq 0, -1 \leq a \leq b \leq 1)$	A$(a + b = -1)$	$(0, 0, 0, -\frac{a^2+b^2+1}{2})$
$A^{ab}_{4,6}$	$e_{14} = ae_1, e_{24} = be_2 - e_3, e_{34} = e_2 + be_3$ $(a \neq 0, b \geq 0)$	A$(a = -2b)$	$(0, 0, 0, \frac{2-a^2-2b^2}{2})$
$A_{4,7}$	$e_{23} = e_1, e_{14} = 2e_2, e_{23} = e_2, e_{34} = e_2 + e_3$	B	$(0, 0, 0, -3)$
$A_{4,8}$	$e_{23} = e_1, e_{24} = e_2, e_{34} = -e_3$	A	$(0, 0, 0, -1)$
$A^b_{4,9}$	$e_{23} = e_1, e_{14} = (1 + b)e_2, e_{24} = e_2$ $e_{34} = be_3\,, (-1 < b \leq 1)$	B	$(0, 0, 0, {\scriptstyle -1-b-b^2})$
$A_{4,10}$	$e_{23} = e_1, e_{24} = -e_3, e_{34} = e_2$	A	$(0, 0, 0, 1)$
$A_{4,11}$	$e_{23} = e_1, e_{14} = 2ae_2, e_{24} = ae_2 - e_3$ $e_{34} = e_2 + ae_3\,, (a > 0)$	B	$(0, 0, 0, 1 - 3a^2)$
$A_{4,12}$	$e_{13} = e_1, e_{23} = e_2, e_{14} = -e_2, e_{24} = e_1$	B	$(0, 0, -1, 1)$

Table 1: The complete list of Lie algebras in dimensions up to four that are not direct sums of lower dimensional ones. The commutator $[e_i, e_j]$ is denoted by e_{ij} for short. The last column gives the Cartan-Killing metric which is diagonal except for $A_{3,8}$ which is minor-diagonal shown by $\{\}$ and represents $SO(2, 1)$.

References

[1] J. Scherk and J. H. Schwarz, Nucl. Phys. B **153** (1979) 61.

[2] H. Nishino and E. Sezgin, "Matter and gauge couplings of N=2 supergravity in six-dimensions", Phys. Lett. B **144** (1984) 187;

H. Nishino and E. Sezgin, "The complete N=2, D = 6 supergravity with matter and Yang-Mills couplings", Nucl. Phys. B **278** (1986) 353.

[3] A. Salam and E. Sezgin, "Chiral compactification on Minkowski $\times S^2$ of N=2 Einstein-Maxwell supergravity in six-dimensions," Phys. Lett. B **147** (1984) 47.

[4] J. J. Halliwell, "Classical and quantum cosmology of the Salam-Sezgin model," Nucl. Phys. B **286** (1987) 729.

[5] K. Maeda and H. Nishino, "Attractor universe in six-dimensional N=2 supergravity Kaluza-Klein theory," Phys. Lett. B **158** (1985) 381.

[6] Y. Aghababaie, C. P. Burgess, S. L. Parameswaran and F. Quevedo, "Towards a naturally small cosmological constant from branes in 6D supergravity," Nucl. Phys. B **680** (2004) 389 [arXiv:hep-th/0304256].

[7] G. W. Gibbons, R. Güven and C. N. Pope, "3-branes and uniqueness of the Salam-Sezgin vacuum," Phys. Lett. B **595** (2004) 498 [arXiv:hep-th/0307238].

[8] V. P. Nair and S. Randjbar-Daemi, "Nonsingular 4d-flat branes in six-dimensional supergravities," JHEP **0503** (2005) 049 [arXiv:hep-th/0408063].

[9] B. M. N. Carter, A. B. Nielsen and D. L. Wiltshire, "Hybrid brane worlds in the Salam-Sezgin model," JHEP **0607** (2006) 034 [arXiv:hep-th/0602086].

[10] S. Randjbar-Daemi, A. Salam, E. Sezgin and J. Strathdee, "An anomaly free model in six-dimensions", Phys. Lett. B **151** (1985) 351.

[11] S. D. Avramis, A. Kehagias and S. Randjbar-Daemi, "A new anomaly-free gauged supergravity in six dimensions", [arXiv:hep-th/0504033].

[12] S. D. Avramis and A. Kehagias, "A systematic search for anomaly-free supergravities in six dimensions", [arXiv:hep-th/0508172].

[13] M. Cvetic, G. W. Gibbons and C. N. Pope, *A string and M-theory origin for the Salam-Sezgin model*, Nucl. Phys. B **677** (2004) 164 [arXiv:hep-th/0308026].

[14] E. Bergshoeff, D. C. Jong and E. Sezgin, "Noncompact gaugings, chiral reduction and dual sigma models in supergravity," Class. Quant. Grav. **23** (2006) 2803 [arXiv:hep-th/0509203].

[15] S.W. Hawking, Mon. Not. R. Astron. Soc. **142** (1969) 129.

[16] E. Bergshoeff, U. Gran, R. Linares, M. Nielsen, T. Ortin and D. Roest, "The Bianchi classification of maximal $D = 8$ gauged supergravities", Class. Quantum. Grav. **20** (2003) 3997 [arXive:hep-th/0306179].

[17] G.M. Mubarakzyanov' Izv. Vysshikh Uchebn. Zavedenii Math. No. **1** (32) 114; No.3 (34) 99; No. 4 (35) 104 (1963).

[18] J. Patera, R. T. Sharp, P. Winternitz and H. Zassenhaus, "Invariants Of Real Low Dimension Lie Algebras," J. Math. Phys. **17** (1976) 986.

[19] J. Maharana and J. H. Schwarz, "Noncompact symmetries in string theory," Nucl. Phys. B **390** (1993) 3 [arXiv:hep-th/9207016].

[20] A. Salam and E. Sezgin, "D = 8 Supergravity: Matter Couplings, Gauging And Minkowski Compactification," Phys. Lett. B **154** (1985) 37.

[21] P. K. Townsend and P. van Nieuwenhuizen, "Gauged Seven-Dimensional Supergravity," Phys. Lett. B **125** (1983) 41.

[22] A. Salam and E. Sezgin, "SO(4) Gauging Of N=2 Supergravity In Seven-Dimensions," Phys. Lett. B **126** (1983) 295.

[23] E. Bergshoeff, I. G. Koh and E. Sezgin, "Yang-Mills / Einstein Supergravity In Seven-Dimensions," Phys. Rev. D **32** (1985) 1353.

[24] L. J. Romans, "The F(4) Gauged Supergravity In Six-Dimensions," Nucl. Phys. B **269** (1986) 691.

[25] L. Andrianopoli, R. D'Auria and S. Vaula, "Matter coupled F(4) gauged supergravity Lagrangian," JHEP **0105** (2001) 065 [arXiv:hep-th/0104155].

[26] G. Dall'Agata, C. Herrmann and M. Zagermann, "General matter coupled N = 4 gauged supergravity in five dimensions," Nucl. Phys. B **612** (2001) 123 [arXiv:hep-th/0103106].

[27] L. J. Romans, "Gauged N=4 Supergravities In Five-Dimensions And Their Magnetovac Backgrounds," Nucl. Phys. B **267** (1986) 433.

[28] M. Awada and P. K. Townsend, "N=4 Maxwell-Einstein Supergravity In Five-Dimensions And Its SU(2) Gauging," Nucl. Phys. B **255** (1985) 617.

[29] E. Bergshoeff, I. G. Koh and E. Sezgin, "Coupling Of Yang-Mills To N=4, D = 4 Supergravity," Phys. Lett. B **155** (1985) 71.

[30] M. de Roo and P. Wagemans, "Gauge Matter Coupling In N=4 Supergravity," Nucl. Phys. B **262** (1985) 644.

[31] J. Schon and M. Weidner, "Gauged N = 4 supergravities," JHEP **0605** (2006) 034 [arXiv:hep-th/0602024].

[32] H. Nicolai and H. Samtleben, "N = 8 matter coupled AdS(3) supergravities," Phys. Lett. B **514** (2001) 165 [arXiv:hep-th/0106153].

[33] A. H. Chamseddine and W. A. Sabra, Phys. Lett. B **476** (2000) 415 rXiv:hep-th/9911180].

[34] A. H. Chamseddine and M. S. Volkov, "Non-Abelian solitons in N = 4 gauged supergravity and leading order string Phys. Rev. D **57** (1998) 6242 [arXiv:hep-th/9711181].

[35] M. de Roo, M. G. C. Eenink, D. B. Westra and S. Panda, "Group manifold reduction of dual N = 1 d = 10 supergravity," JHEP **0506** (2005) 077 [arXiv:hep-th/0503059].

[36] E. Bergshoeff, M. de Roo and E. Eyras, "Gauged supergravity from dimensional reduction," Phys. Lett. B **413** (1997) 70 [arXiv:hep-th/9707130].

[37] E. Bergshoeff, M. de Roo, B. de Wit and P. van Nieuwenhuizen, "Ten-Dimensional Maxwell-Einstein Supergravity, Its Currents, And The Issue Of Its Auxiliary Fields," Nucl. Phys. B **195** (1982) 97.

[38] G. F. Chapline and N. S. Manton, "Unification Of Yang-Mills Theory And Supergravity In Ten-Dimensions," Phys. Lett. B **120**, 105 (1983).

[39] M. B. Green and J. H. Schwarz, "Anomaly Cancellation In Supersymmetric D=10 Gauge Theory And Superstring Theory," Phys. Lett. B **149**, 117 (1984).

[40] M. Cvetic, G. W. Gibbons, H. Lu and C. N. Pope, "Consistent group and coset reductions of the bosonic string," Class. Quant. Grav. **20**, 5161 (2003) [arXiv:hep-th/0306043].

[41] A. Salam and E. Sezgin, "D = 8 Supergravity," Nucl. Phys. B **258**, 284 (1985).